Arithmetic Sequences

5.6

***MathLinks 10: Pathways to Success*, pages 264–279**

Suggested Timing
100–120 min

Materials
- grid paper
- graphing calculator or spreadsheet software

Blackline Masters
BLM 5–2 Chapter 5 Warm-Up
BLM 5–14 Staircase Numbers Tables
BLM 5–15 Section 5.6 Extra Practice
BLM 5–16 Section 5.6 Competency Check #11
Master 2 Curricular Competency Rubric
TM 5–1 5.6 Example 1b) Using TI-Nspire
TM 5–2 5.6 Example 1b) Using TI83/84

Big Idea and Elaborations

Constant rate of change is an essential attribute of linear relations and has meaning in different representations and contexts.

Representing and analyzing situations allows us to notice and wonder about relationships.

In this section, students will focus on
- deriving a rule for determining the general term of an arithmetic sequence
- determining t_1, d, n, or t_n in a problem that involves an arithmetic sequence
- describing the relationship between an arithmetic sequence and a linear function
- solving a problem that involves an arithmetic sequence

Planning Notes

Have students complete the warm-up questions on **BLM 5–2 Chapter 5 Warm-Up** to reinforce prerequisite skills for this section.

Discuss Halley's Comet and see if students can determine the next time the comet will be visible from Earth. Ask questions such as the following:
- How many years passed from the first recorded comet sighting to the second sighting?
- What about from the second sighting to the third sighting?
- How did Halley determine that the next time the comet would appear would be in 1758?
- Can you predict the next time that Halley's Comet will appear?

Tell students that by the end of this activity, they will be able to predict the next appearance of the comet and develop a sequence indicating in which years the comet will appear in the next 1000 years.

Explore and Analyze

The staircase example gives students an opportunity to examine a number of different arithmetic sequences and provides a concrete method of generating these sequences. The goal of the Explore and Analyze is to help students determine whether the tables represent a linear or a non-linear relationship. Students should work in pairs to complete the Explore and Analyze and then share their findings with the class. As you summarize as a class, you may wish to open up the discussion to the following topics:
- Are there any other types of sequences?
- If arithmetic sequences have a rule that consecutive terms have a common difference, can you think of a way to produce a set of sequences using a similar but different rule? Explain.
- Produce three different sequences using your rule.

Ask the following questions to help students understand the two-step staircase example:
- How many cubes are used to build the first two steps?
- How many cubes are used to build the second and third steps?
- How many cubes are used to build the third and fourth steps?

Have students fill in the remaining cells in the table and list all the values as a sequence of numbers. Ask if they notice any special attributes that determine how each term is related to the next term and to the previous term.

For the two-step staircase example, some students may have difficulty seeing the pattern. Remind them that these are very similar to patterning questions they have been doing since elementary school. Ask them how

they have solved them in the past without a formula. They may benefit from the following leading questions:

- What pattern do you see in the sequence 3, 5, 7, 9, 11, 13, 15, 17, 19, 21?
- What is the first term?
- What is the difference between terms?
- How many terms are there?
- Can you think of a way to represent each term using its term number? Explain.
- Can you write the value of the term next to its term number? Explain. What are the term numbers of 17, 19, and 21?

For #6 to #8, have students explain their strategy. Prompt them with the following questions:

- Which features of the sequences are similar to the two-step staircase: first term? difference between terms? number of terms?
- Which features are different?
- Which feature defines the sequences as being arithmetic?

Suggest to students that they use the two-step staircase as a model to complete the three-step staircase tables. Then, have them describe their strategy for determining the number of cubes in each staircase.

For #9, students are required to see the relationship between the term number and the number of differences from the first term. If they are having difficulty, have them use a table like this:

Term #	t_1	t_2	t_3	t_4	t_n
1	3	5	7	9	
2	3	3 + 2	3 + 2 + 2	3 + 2 + 2 + 2	
3	3 + (0)2	3 + (1)2	3 + (2)2	3 + (3)2	

Have them fill in columns for t_5 to t_7. Prompt their thinking with questions such as the following:

- Do you see any relationship between the term number and the number of differences you add? Explain.
- If there are 100 terms, how many differences would you add to the first term?
- Determine the 100th term. How can you express this relationship using n as the variable for the term number?

Differentiated Instruction

- Have students perform the operations to determine the sequence in the opening paragraph. Discuss why the interval between sightings of Halley's Comet is not always 76 years. It is important for students to understand that the numbers are a description of the natural world and that sometimes there are anomalies in the natural world. Ask if these concepts relate to concepts being studied in science.
- For the Explore and Analyze, have students make the staircases with blocks or cubes, or draw the staircases and count the blocks.
- Have students use **BLM 5–14 Staircase Numbers Tables** to complete the tables in the Explore and Analyze.
- It may be beneficial for students to work with a partner through the Explore and Analyze.
- The nth term is very abstract for students. Ask the class why or when would you want to know about the nth term.
- Tell students that an arithmetic sequence is a list of numbers with a common difference. Challenge them to create a way of testing the following hypotheses:
  - The sum of two arithmetic sequences is another arithmetic sequence.
  - The product of two arithmetic sequences is another arithmetic sequence.

Suggest that students test each hypothesis using arithmetic sequences. Then, have them test with another pair of sequences. If the hypothesis appears to be true, ask them to use a general case to summarize their findings.

ELL

- Make sure students make the connection between the photo of Halley's Comet in the student resource and the term *Halley's Comet*.

Common Misconceptions

- Some students may have difficulty understanding that arithmetic sequences have a common difference between terms.

R They should get in the habit of using at least three sets of consecutive terms to determine the common difference.

- Some students may not be able to see that any term in the Explore and Analyze can be determined by adding the first term and a product of one less than the term number and the common difference.

R Lead them to see the relationship by using one-on-one discussions about the connection between the sum of the first term and one less than the term number. A review of which words relate to each operation may be helpful. For example, you might ask:

- If you use n to represent the term number, how would you write one less than the term number?
- How would you represent the product of one less than the term number and the common difference?
- If you use t_1 to represent the first number, how will you write the sum of the first term and the product of one less than the term number and the common difference?

- Some students may not understand how to use subscripts to define specific terms in the sequence.

R Give students practice by, for example, asking them to represent the fifth term of a sequence using t to represent a term and a subscript to represent the term number.

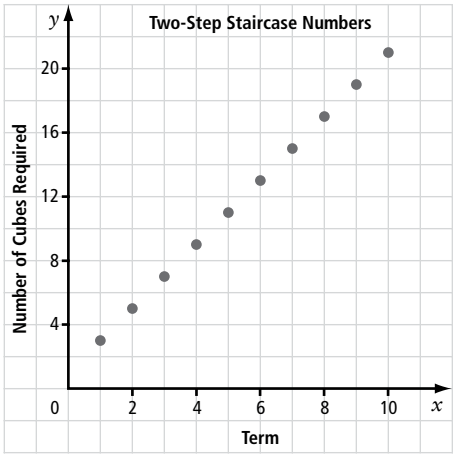
Explore and Analyze Answers

1.

Term	1	2	3	4	5	6	7	8	9	10
Staircase Number (Number of Cubes Required)	3	5	7	9	11	13	15	17	19	21

2. a) Linear. The difference between each staircase number is a constant, 2.

b)



Discrete

3. a) Add 2 cubes to get each next value.

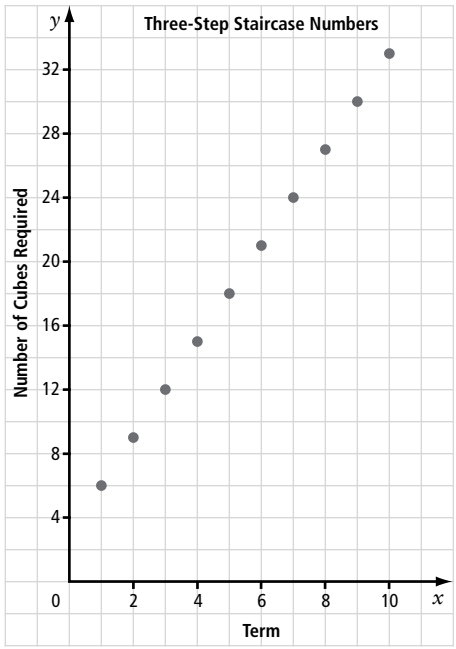
b) To find the 11th term, add 2 to the 10th term. To find the 12th term, add 2 to the 11th term.

4.

Term	1	2	3	4	5	6	7	8	9	10
Staircase Number (Number of Cubes Required)	6	9	12	15	18	21	24	27	30	33

5. a) Linear. The difference between each term is a constant, 3.

b)



Discrete

6. a) Add 3 cubes to get each next value.

b) To find the 11th term, add 3 to the 10th term. To find the 12th term, add 3 to the 11th term.

7. a) Yes. They form a list of numbers that are in order.

b) All pairs of consecutive terms have a common difference.

8. a) Add a value equal to the number of steps.

b) Yes.

9. a) Determine a pattern; then develop a formula to represent the pattern using a variable to represent the number of terms.

b) $t_{100} = 3 + (n - 1)2$
$= 3 + (100 - 1)2$
$= 3 + (99)(2)$
$= 3 + 198$
$= 201$

c) $t_n = t_1 + (n - 1)d$, where t_1 is the first term, n is the number of terms, and d is the difference between terms

Assessment Opportunity	Supported Learning
Assessment *as* Learning	
Reflect and Respond	• Encourage students to use the Explore and Analyze to assist them in answering the questions. • Suggest that students use more than one method when answering #3b) and #6b). They could compare their strategies with another student or group. Provide manipulatives (such as linking cubes) where possible. • If students are still struggling, complete Example 1 as a class and then ask them to return to the Reflect and Respond questions. To assess understanding, ask them for the 50th term. • Discuss #9 as a class to allow students to hear multiple strategies and reasoning for the solutions.

Develop Understanding

With the class, discuss the definitions. Discuss which terms students are already familiar with and which ones are new to them.

Have students consider the sample sequence 10, 16, 22, 28, …. Ask them what relationship they notice between the term number and the number of common differences added to that term. If they have problems answering this question, have them refer to the following list in the student resource and note what is added in each expression:

$t_1 = t_1$ Add 0.
$t_2 = t_1 + d$ Add d.
$t_3 = t_1 + 2d$ Add $2d$.
⋮
$t_n = t_1 + (n - 1)d$ Add $(n - 1)d$.

Another approach to doing these types of questions is to use logic. For example, students can look at the term number and the number of common differences. Ensure students understand why n is restricted to the natural numbers.

Example 1 Notes

In this example, students are given at least three terms and the nth term of the sequence and are asked to identify the number of terms in the sequence.

Encourage students to continue the habit of using t_1 to represent the first term and d to represent the common difference. You may wish to guide them to understand that they can determine the common difference by subtracting any three consecutive terms (for example, $d = t_4 - t_3$, $d = t_3 - t_2$, and $d = t_2 - t_1$), and that if the values are the same, the sequence is arithmetic. They can then write the formula $t_n = t_1 + (n - 1)d$. By substituting for t_1, d, and t_n, they can solve for n, the term number or number of years in this case.

Students may benefit from a one-on-one or small-group discussion in which they share how they determined the first term, the common difference, and the nth term of the sequence given. Then, they share how they would use the given data to solve for n.

You might have students explore how technology can be used to support their answers. Examples of a graphing calculator using formulas with list names and a spreadsheet are shown in the example. You may wish to have students use **TM 5–1 5.6 Example 1b) Using TI-Nspire** or **TM 5-2 5.6 Example 1b) Using TI83/84**.

You may wish to approach the Your Turn as a class. The question involves a sequence for the height of a child between the ages of 3 and 10. Since the starting age is not 1, students can decide whether the sequence should start with $t_1 = 95$ or $t_3 = 95$. Their decision will affect the formula for the general term in part a) and the value of n used in part b).

Example 2 Notes

Encourage students to get into the habit of using t_1 to represent the first term, using d to represent the common difference, and then writing the general formula. By substitution, they should then be able to determine the general term. This strategy will assist them with any sequence or series question that they need to solve by helping them determine which formulas they can use. It also allows them to get a concrete and visual understanding of the relationship between the terms of a sequence and any given term or the sum of a series. Encourage them to practise this strategy with the Your Turn question for Example 1 and the Connect and Reflect questions.

Encourage students to use the same process for the Your Turn by listing known values and using the general formula to solve for the term number.

Example 3 Notes

This example requires students to list a set of terms of a sequence using a description. Then, they are asked to find the general term using the given information. Finally, they are asked to find the 10th term by using the general term or by using technology to graph the sequence.

Method 2 of part c) illustrates a graph drawn using a spreadsheet. You may wish to refer to **TM 5–1 5.6 Example 1 Using TI-Nspire**, and **TM 5–2 5.6 Example 1 Using TI83/84,** which show students how to make graphs using technology.

Note that the graph in the student resource shows a cost of $65 when the number of hours worked is zero. You may wish to discuss what this amount means.

Cultural Connection

For Example 1, you may wish to have students research arts groups or centres that are relevant to their community or culture. For Example 2, consider having them research the importance of the muskox and caribou to Aboriginal groups. For example, some Inuit use the muskox's soft underwool, called *qiviut* (pronounced kiv-ee-ut), for weaving shawls, sweaters, gloves, hats, scarves, and other items.

Differentiated Instruction

- When developing the formula to determine the general term of an arithmetic sequence, be sure that students understand what t_n, t_1, n, and d represent. Post several examples of arithmetic sequences, and have students point out the various parts. Work through t_2, t_5, and so on, to check that students understand what these variables represent.
- You may wish to demonstrate Example 3c), Method 2, on a projector with a graphing calculator or a similar computer-generated graph. Students will benefit from the visual representation of the solution.

ELL

- Some students may need assistance in understanding the terms *performing arts group* and *community events leader* in Example 1. It may help if you break down the terms into individual words to guide students to what the composite terms mean.
- Some students will not be familiar with muskox or caribou. Show them pictures of each animal as you repeat the names.
- Show students who do not know the term *furnace technician* in Example 3 a picture, and describe what tasks this job involves.
- Make sure students include the following in their vocabulary dictionaries: *arithmetic sequence*, *term of a sequence*, *general term*, *finite sequence*, *infinite sequence*, *common difference*, *first term*, *number of terms*, and *nth term*. Encourage them to include verbal descriptions, diagrams, and/or examples. Also have them include the variables t_1, n, d, and t_n and what they represent.

Common Misconceptions

- Some students may find it challenging to identify the variables for the equation.

℞ Encourage them to develop a habit of writing down the known information by listing t_1, d, and/or n, and then writing the equation for t_n. Suggest that they always identify the target or goal of each question by indicating which variable is unknown, for example, $n = ?$ They can then substitute the known data into the equation and solve for the unknown value.

- Some students may find the structure of the general term formula confusing with d appearing after the brackets, $(n - 1)d$.

℞ Remind them that multiplication is commutative.

- Some students may not recognize a sequence as arithmetic.

℞ They should identify the difference for at least three pairs of consecutive terms. This practice will help them to see the pattern.

Your Turn Answers

Complete solutions to the Your Turns appear in the respective Chapter Solutions.

Example 1: Your Turn

a) $t_n = 95 + (n - 1)5$, where n is the number of years after year 2, or $t_n = 95 + 5(n - 3)$, where n is the child's age, $n > 2$

b) 130 cm

Example 2: Your Turn

37 months

Example 3: Your Turn

$505

Assessment Opportunity	Supported Learning
Assessment *as* Learning	
Example 1	• Students may benefit from verbalizing and writing out the sequence. • Have students work in pairs when working with technology to support each other's work.
Assessment *for* Learning	
Your Turn	• Students may benefit from verbalizing and writing out the sequence so that it is visually available to them. • Review the meaning of the variables in the equation. • Encourage students to always list the known and unknown values so that they can visualize what they are solving for. Also have them write out the general formula before they begin substituting values. • Allow calculator work only after you are certain students understand what they are solving for and can estimate an answer.

Assessment Opportunity	Supported Learning
Assessment *as* Learning	
Example 2	• Have students list the known and unknown values. • Remind students that the methods for solving equations are the same regardless of the equation. • You may wish to have students complete an additional calculation for Example 2 before moving onto the Your Turn question.
Assessment *for* Learning	
Your Turn	• Have students list the next three values in the sequence so they are engaged in understanding and generating numbers to fit a pattern. • Have students verbally describe what they are trying to find. • Have students list the known and unknown values.
Assessment *as* Learning	
Example 3	• Have students work in pairs to discuss the advantages and disadvantages of each method.
Assessment *for* Learning	
Your Turn	• Review what it means to find a general term. • Encourage students to record the known and unknown values. • Writing part of the sequence may assist visual learners.

Connect and Reflect

Questions Competency Guide

	Question Number		
	Essential	Typical	Extension/Enrichment
Reasoning and Modelling	4, 5	8, 10, 15	18, 19, 21
Understanding and Solving	1–7, 9	8, 10–16	17, 18
Communicating and Representing	7	8, 11, 13–16	17–21
Connecting and Reflecting	4–7	11, 12, 16	17, 18, 21

Key Ideas Notes

Students may benefit from writing down the Key Ideas in their own words before completing the Connect and Reflect questions. They should get into the habit of using the Key Ideas as a reminder of what skills and knowledge are most important in each section. If they write down these key concepts, they will always have a quick reference when needed.

Practise, Apply, Extend Notes

Students will find #1 to #7 to be very similar to the examples in the student resource. Encourage them to practise writing down known information from the question. They can then use the equation they formulate, when appropriate, to solve for missing values (t_1 or d) or unknown terms.

For #5, students need to recognize that they have been given enough terms to determine t_1 and d. You may wish to ask them leading questions:

- What is the first term? What variable do we use to represent the first term?
- How can you determine the common difference?
- Using the first term, t_1; the common difference, d; and the nth term, can you substitute these values into the general equation and solve for n? Explain.

For #7, you may need to meet with students individually or in small groups to discuss how they can use a graph to identify the terms of an arithmetic sequence. Have students list the ordered pairs shown on the graph. Then, guide them with prompts:

- What do you notice about the values of the first terms of the ordered pairs?
- Which number system do they represent?
- In arithmetic sequences, which number system do the term numbers represent?

- From the list of ordered pairs, do the x-values or y-values represent the values of the sequence?
- What is the first term?
- What is the common difference?
- How many terms are there?
- Can you express this sequence using a general term? Explain.

Students need to recognize that arithmetic sequences, when graphed, form discrete linear graphs. They also need to recognize the similarity between the general term of the sequence and a linear relation equation of the form $y = ax + b$ from grade 9. You may wish to use $y = mx + b$, but they are not introduced to the Key Term *slope-intercept form* until Chapter 6 Linear Equations and Graphs. Have them write down the general term of the given sequence and then the slope-intercept form of a linear function. They may require prompts to help them:

- Do you remember how to express the equation of a linear relation?
- What does a (or m) represent?
- What does b represent?
- Determine and simplify the general term of the sequence given in the graph. Does your general term look similar to a linear equation?
- What is the rate of change/slope from your general term? What is the y-intercept?

For #8, students need to recognize that the value of n must be a whole number. If they use the given information to solve for n and determine that n is not a whole number, they can assume that the given term (34) is not a part of the sequence. Instead of giving students this information directly, guide them toward this understanding. Ask:

- Which number system have we been using for the values of n in the general term?
- Solve for n in part A of the question. Is n a whole number?
- Is 34 a term of this sequence? Explain.
- What is the term number for 34?
- Repeat these steps for parts B to D. Did you determine any solutions for n where n is not a whole number?
- If n is not a whole number, do you think that 34 is a term of this sequence? Why or why not?

For #11, students who have golfed are more likely to make connections with the context of the situation. You may wish to ask these students to work with students who have not golfed. While solving, students perform operations using time values. You may need to remind them that these expressions are not decimal values and that the next value after 8:59 a.m. is 9:00 a.m. not 8:60 a.m. Some students may be more comfortable working with fractional times allowing them to review some previous learning and make additional connections. Students should use their calculated values for part d) to justify the factors identified in part e). Ask students if it is more likely to have an early or delayed tee-off time.

For #12, students follow a similar process as in previous questions. Assist them by using the following prompts:

- What is the area of the artwork?
- What term of an arithmetic sequence does the area represent?
- Which variable of the general equation represents the work completed on the first day?
- If the progress on the artwork forms an arithmetic sequence, what variable from the general equation represents the work completed each day?
- Are there pieces of art in the school that can be similarly analyzed?

For #13, students must make a connection between the variables in the general equation of an arithmetic sequence and the number of carbon atoms or hydrogen atoms. Use leading questions to assist students with their thinking:

- Given the table, which variable, t_1, d, n, or t_n, represents the number of carbon atoms?
- Which variable represents the number of hydrogen atoms?

Once students make this connection, they should be able to list the variables and determine the general term.

The next question, #14, requires students to identify terms within a certain range of values. To help students determine the nth term within the range, encourage them to use multiples of 28. Ask them the following questions:

- What is the largest multiple of 28 that is less than 1000?
- How did you determine this value?

To assist students in determining the first term within the given range of the second column, have them use multiples of 7. Use the following prompts:

- What is the smallest number greater than 500 that is divisible by 7?
- If you begin a new sequence starting at 504, what are the next three terms of the sequence?
- What is the largest number divisible by 7 that is less that 600?
- Which variable, t_1, d, n, *or* t_n, does this value represent?

Students should then be able to apply their new skills to complete the table.

You may wish to have students work on #15 in pairs or small groups. Have them work together to recognize a relationship between the change in pressure and each metre of descent. Use leading questions to guide students:

- If a 10 m descent produces a change of 101 kPa, what pressure change occurs for a descent of 1 m?
- What is the atmospheric pressure at sea level?
- What is the change in pressure for each metre of descent?
- What variable will you use to represent the pressure at sea level?
- What variable will you use to represent the total change in descent?

Students should then use their knowledge of arithmetic sequences and linear graphs to answer the question.

It may be beneficial for students to draw a diagram for #17 to help them recognize how the radius increases with each support tower.

For #18, encourage students to consider using a 24 h clock. Discuss how operations involving time values are different from operations involving decimal numbers. Have them share how these operations differ.

Create Connections Notes

Make an enlarged copy of #19 for each student. Have them cut out the squares with symbols inside and place them on top of the given answer squares. Ask them to write one or two questions of their own that could be answered using the symbols.

For #20, suggest that students use their graphic organizers to summarize their understanding of arithmetic sequences.

The Mini Lab in #21 can be used as an interactive activity for students to see what happens to graphs when the first term changes and the common difference changes. This activity will help them to see the relationship between the first term and the y-intercept, and between the common difference and the slope of the graph. Have them work in pairs or groups.

Cultural Connection

For #12, suggest that students research other artists of Inuit wall hangings. You might have them write their own problems based on the wall hangings they find during their research. They can then solve each other's problems.

Related to #15, you may wish to discuss the beluga whales' connection to Inuit communities. Because of their importance in Inuit culture, beluga whales are a popular subject in Inuit art. Although hunters are discouraged from harvesting female belugas and calves, beluga whales are a source of nutrition and an important part of the Inuit diet. Traditionally, the whale fat was used as fuel for lamps, for heating, and in cooking. Today, whale meat is enjoyed in a number of ways.

Differentiated Instruction

- Provide **BLM 5–15 Section 5.6 Extra Practice** to students who would benefit from more practice.
- Some students will find it difficult to work with the formula. Provide them with structured steps so they can follow along and know what to substitute and when.
- Have a class discussion about #6. Ensure that students understand how to determine the common difference.
- For #7d), some students may need assistance to determine the slope of a discrete linear relation graph.
- Students may wish to have square tiles available to assist with #10. Alternatively, grid paper may be helpful for visual learners.
- For #13, you may wish to have students find out more alkane names.

ELL

- For #11, if students are not familiar with the sport of golf, have another student who is explain the game and what *tee-off time* is.
- The language in #13 and #17 may be challenging to some students. Consider not assigning these questions to English language learners or have them work with a partner who can help them understand the questions.
- Some students may not be familiar with the following terms: *beluga whale*, *water pressure*, *rotation*, *solar eclipse*, *phase*, and *symbols*. Use a combination of descriptions, examples, and pictures to assist in student understanding.

Competency Check Sample Solutions

Question #11

a) Since 8:00 a.m. is considered to be time 0, then $t_1 = 0$. Since the tee-off times are 8 min apart, then $d = 8$. The sequence is 0, 8, 16, 24.

b) Extend the sequence to 60 min.
0, 8, 16, 24, 32, 40, 48, 56, …
So within the first hour, 8 groups of four will have teed off. This means 32 players will be on the course after 1 h.

c) $t_n = 0 + (n - 1)8$
$= 8n - 8$

d) For 156 players, there will need to be 156 ÷ 4 or 39 groups teeing off.
Substitute $n = 39$ into t_n.
$t_n = 8n - 8$
$= 8(39) - 8$
$= 312 - 8$
$= 304$
This means that the last group will tee off 304 min after the first group.
304 min = 5 h 4 min, so the last group will tee off at 8:00 + 5:04, or 1:04 p.m.

e) Example: Rain may interrupt the tee-off times or players in a group might not be quite ready at their tee-off time.

You may also choose to use the Competency Check question as a summative assessment. See **BLM 5–16 Section 5.6 Competency Check #11** and **Master 2 Curricular Competency Rubric**.

Assessment Opportunity	Supported Learning
Assessment *for* Learning	
Practise and Apply	• It may benefit some students to compare their responses with a partner's. • For #6, students may use the work they completed in the Your Turns or from the examples as a quick review. • For #9, some students could be encouraged to write down 16 dashed lines and fill in the 16th number. From here, they could work to the left by subtracting 7. This may help them understand the sequence initially, but encourage them to use the formula to solve for the same first value. • Encourage visual learners to always list some of the numbers from a sequence. You might provide a guideline for when it is acceptable to write out a sequence to find a term and when it is not (the length of the sequence being the main deciding factor). • For #18, review the 24 h clock with students.
Assessment *as* Learning	
Create Connections	• To assist students with #19, have them create their own arithmetic sequence with a positive value for d and then a negative value for d. They should identify the values for t_1, d, and n. Have them refer to this example to select the responses from the list. • Encourage students to write their own descriptions, definitions, and examples for the graphic organizer in #20. Use their work to assess their understanding.